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On Cantorian spacetime over number systems with division by zero

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Abstract

Division by zero is proposed in order to open new horizons for abstract mathematics and physics. Although the formerly forbidden operation may be of little practical value, the feasibility of unrestricted operations is needed for meaningful deployment of linear vector spaces, whose duals are inverse transposes of their primary vector spaces. Both quantum mechanics and pan-geometry of multispatial hyperspace operate on vectors that belong to mutually dual linear vector spaces. The division by zero also may remove many years old paradox related to multiplication of zeros and makes it possible to introduce the concept of bigroup as an additive and multiplicative group at the same time. This operational extension to algebraic groups makes thus an infinite-dimensional, fractal Cantorian spacetime the preferred, invertible, operationally unrestricted, abstract mathematical infrastructure for physics.

1. Physics needs new mathematical foundations

Some problems of physics could be traced to hidden, unresolved issues in pure mathematics (PM), some of which are almost as old as the, allegedly impossible and therefore prohibited, division by zero. There is really no need and no compelling reason for the prohibition, however. Quite on the contrary, we actually need the banned operation in order for us to advance physics. We have no practical need to divide by zero, but we need to have the freedom to operate on all number fields without any artificially posted restrictions. The prohibition is not just an inconsequential *faux pas*, but an attempt at hiding a part of mathematical (and physical) reality, which might have been misunderstood. Einstein once wondered why such a free creation so well depicts objective physical reality. Mathematics is not a free creation of human mind, however, but a model of nature's operational side. As long as it is kept in sync with the nature, it does depict the reality. Once its touch with the reality is lost, it becomes a pointless art. This happened to the PM.

The quest for operationally unrestricted mathematics is oftentimes tied to issues of abstract mathematical existence. Although an abstract existence in the PM used to mean being well defined and thus also noncontradictory [1], there is no assurance that an old concept will not cause any new problems in a new theoretical context. Historically, mathematics was presented as most exact, splendidly abstract and highly analytic science, whose theorems can be decomposed into linear sequences of strict derivations. Yet it was also an inherently synthetic science, built upon constructions that are based on a few primitive notions and on formal derivations from them [2]. We used to think that mathematical discovery has obtained somewhat permanent and almost universal validity [3]. Since even the actual meaning and the scope of most concepts is learned through their practical use [4], there is a good chance that a newly acquired meaning, or some extensions thereof, may not be quite compatible with the concept's presumed traditional meaning. The need for

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reevaluation of mathematics does not always mean that what was proved becomes invalid. It means that boundaries of validity may be quite different after discoveries of new facts. Analysis itself is thus insufficient for predictions of entirely new phenomena. It should be complemented by an essentially empirical synthesis. PM missed the synthesis almost entirely.

Discovery of nonEuclidean geometries, for example, did not invalidate the Euclidean one. Nevertheless, it has eradicated a large part of previous pure-mathematical activities that aimed at defending the Euclid's odd 5th postulate. Although it did not change the Euclidean geometry, it changed our thinking. Because mathematics provides abstract language for physics, and physical reality cannot be deduced from a priori philosophical concepts [5], therefore mathematics may need an upgrade after a discovery is made in physics, for some old ideas may be irrelevant to new aspects of the physical reality. We need a synthetic mathematics (SM) to complement the classical analytic methods in mathematical research. Induction alone is insufficient for the syntheses needed to comprehend physics. Yet some new aspects of the physical reality could be deduced from experiments backed by the SM.

Its inductive abstraction and strict rules of inference for deduction made mathematics the most exact of all exact sciences. Yet the PM emphasized its apriorical character and its "statutory" independence of any experimental evidence almost to the point of self-destruction. For by allowing the use of postulative method to define its primitive notions and fundamental objects, its splendid exactness is practically defeated. Although formal construction was supposed to alleviate the arbitrariness of its postulates, there is no way to pinpoint faulty constructions if they are formally quite correct. For if one can postulate attributes of some building blocks ("bricks") in their formally noncontradictory definitions, these bricks acquire an abstract mathematical existence, even though they may imply some paradoxical consequences at a deeper level. Hence one could still build quite nonexistent abstract building with nonexistent bricks. Therefore we should try to discover all properties of those abstract fundamental objects through formal syntheses from logical requirements for their existence, and from those abstract logical constraints that would post formal limits on their existence. In particular, I should not postulate any features of zero or infinity, but to discover them through exact operations on abstract operational structures, which they are supposed to fit.

The celebrated highly formalized approach to mathematics has actually blinded us so that we do not see all the relationships between mathematical objects, because we were not supposed to see them thanks to their allegedly perfect formalization. Many physical experiments remain unexplained even though plausible explanations could have been conjectured, were it not for the fact that such hypothetical explanations would contradict certain a priori assumed mathematical ideas, which have been uncritically transplanted into mathematical physics (MP). Historically, oftentimes we were not haunted by our inability to solve problems, as we tend to think, but by our ability to get correct results from incomplete premises. Newton hesitated for over 20 years before he published his Principia, but it was not because he got wrong results. As a matter of fact, planets confirmed his predictions to great extent even though he knew, I think, that his mathematical assessment of gravity was somewhat incomplete [6]. The very fact that he got quite correct results from not so unquestionable assumptions raised his doubts that caused him to postpone publication of his great Principia, I suppose. Although his doubts were presumably related to physics, their roots were of mathematical nature [6]. We should therefore get rid of old mathematical prejudices and expose fundamental flaws of some traditional mathematical concepts [7] in order to develop new physics, which is severely hampered by obsolete mathematics.

Mathematical treatment of time flow requires a new geometric approach. Physical time seems to flow in three time-based abstract dimensions [8–10], but such a conclusion was too far-fetched for the former PM and MP. Lack of abstract mathematical tools for adequate synthetic modeling of physical phenomena deferred the development of physical theories, which lag behind experiments. Many results of recent experiments and observations remain unexplained, because the PM still operates within the perimeter outlined by some ancient and medieval paradigms. Therefore we must upgrade abstract mathematics after significant breakthroughs in physics. It is not enough to justify former physical achievements by showing that mathematics complies with them. We should create quite new mathematics that goes far beyond and above of what past physics may have suggested. It is imperative thus to keep mathematics in sync with developments of new ideas in physics.

2. Bigroups as superimposed double groups

The set $\mathfrak{R}$ of real numbers in which for every ordered pair of numbers a binary operation with unique result is defined, is called groupoid [11]. If the operation is associative: $(ab)c = a(bc)$ for arbitrary elements $a, b, c \in \mathfrak{R}$, then we call the set semigroup. When also an inverse operation $I()$ with a unique solution always exists, then we call it group. Hence the groupoids, semigroups and groups are abstract, formal algebraic structures defined over the set of real numbers. Let a group G imposed on the set $\mathfrak{R}$ be denoted by $G := \mathrm{Group}\{\mathfrak{R}(*)\}$, where $\mathfrak{R}$ is the set of elements being operated on and the asterisk is the abstract operation for which the group is defined. The group G must have a unique unit (or neutral element) e such that $a^*e = e^*a = a$ for every $a \in \Re$ [11]. Evidently thus both addition and multiplication of real numbers are proper group operations so that one can speak of additive and multiplicative groups. Henceforth 0 denotes the additive unit and 1 is the multiplicative unit. Now let $I(a) = a^{-1}$ defines the multiplicative inverse $I()$, whereas $R(a) = -a$ shall define additive inverse (i.e., reverse) operation $R()$.

By number system (NS) I call an extended set $\{A \cup \Upsilon\}$ equipped with set of external elements Υ and set A, that would assure unrestricted operations on the extended set. Group $G := \mathrm{Group}\{A \cup \Upsilon(*)\}$ defined on a NS is thus an operational structure as well an abstract algebraic structure. The difference is that in algebraic structures all operations are tailored to fit sets, whereas in operational structures sets should be extended to fit all operations in order to achieve seamless operability. Former theory of linear vector spaces (LVSs) had no meaningful vector multiplication of vectors, for example, until I have introduced it in [7]. The primary reason for this requirement of unrestricted operability is that most algebraic structures were usually only postulated, but rarely constructed, if ever. For physical applications we need constructible operational and algebraic structures without postulative axiomatic crutches. Nature does not care about proofs or derivations, but operability is a must.

Obviously even some well-defined abstract algebraic structures may be operationally incomplete. Although one can talk about postulated things, sometimes one cannot construct physically relevant and logically consistent models founded upon such postulated notions. If postulated structures are not constructed or synthesized, they can sometimes enjoy unreal existence, for postulates just cannot ensure that such postulated objects will be actually noncontradictory. Without mandatory construction and some experimental relevance the required noncontradiction just cannot really be enforced.

The elements of Υ are not extensions of algebraic structures as defined in [12]. My idea is to add few elements as operands to the set A—such as an extra zero and infinity—to set the record straight, rather than to restrict the operational structure of the group itself. This was the main problem of pure mathematics: Instead of defining operations it postulated separate structure for each multioperational entity and by doing so it has conveniently evaded actual constructions. Of course, the PM implicitly postulates that field is a generalization of both additive and multiplicative groups, but never proves it. Fields embrace groups by (postulative) decree, of course, but as you shall see in what follows, there is no way to actually construct a field from groups without creating a paradox. The infamous prohibition on division by zero tacitly covered up the whole problem, without ever removing the paradox. It has merely suppressed any inquiry into possible causes of the paradox.

It is operation that is observed in nature, not a structure. As meaningful as they may be, algebraic structures are our inventions. Such structures are conceptual images of our perceptions of the operational structures that are implemented in nature. Therefore abstract structures should be defined also in terms of actual operations and thus become constructible. Nature never complains about division by zero. Our abstract structures should comply with all operations performed on them. On the other hand, however, group operations determine abstract structures. Hence pure mathematics is not incorrect, but its way of doing things is very convoluted. Therefore instead of starting with algebraic structures and their dimensions, I shall begin with binary operations and their hierarchy using only the real numbers at first.

Generalizing operations for linear vector spaces would make the theory more elegant [13–15], but it would force me to define vectors, whose nature I intend to discover. If I would define vectors right up front, then I would be essentially investigating my present state of mind, and thus reduce the abstract mathematics to a kind of psychopathology, which is what former PM was really very good at. I have already shown that there is much more to vectors than we ever realized [7,8,10] and therefore one should not really postulate anything about them arbitrarily. Neither should I cleverly impose group operations on quaternions treated as just 3D objects [16]. I cannot accept PM's methodology, because it admits a priori postulates. Since I want to discover the meaning of vectors and quaternions, which shall be done elsewhere, therefore I must not define their properties in advance. I should synthesize their properties from abstract operations performed on them. LVSs are extremely important, but postulating their properties can prevent us from learning something new about them, not to mention proper understanding of their full meaning. I am not against making introductory definitions. But I must not use definitions to postulate properties, and then reject some vectors, for instance, because they just do not fit my definition. Such an act could reject a part of the actual reality for no scientific reason.

My goal in this note is to show that the age-old prohibition on division by zero was unnecessary. One could divide by zero, not in order to obtain some fancy results, but to achieve full operational consistency. In the nature there are no separate additive and multiplicative groups with distinct sets of numbers, but all operations are performed on single operational structure, as if nature operates on bigroups or multigroups, in general. Physics should deploy models that match the nature's mode of operations, rather than some pure-mathematical abstract models devised with open disregard for physical reality in order to ease proofs of their postulated properties. If our physical concepts appear less general than those arbitrarily postulated by PM, then perhaps there is a good reason for that, which may still be far beyond our comprehension. Our idea of generality was often historical, not operational.

Let me define now the bigroup B as a double operational structure that is both additive and multiplicative group over the same set at the same time:

$$B \equiv \mathrm{BiGroup}\{\Re \cup \Upsilon(\otimes, \oplus)\} := \mathrm{Group}\{\Re \cup \Upsilon(\otimes)\}\&\mathrm{Group}\{\Re \cup \Upsilon(\oplus)\}. \tag{1}$$

The sequence of the two group operations in the above definition indicates also their priority of execution. Defining groups and bigroups as abstract operational structures rather than just sets with some operations on the set's elements is a slight departure from the standard pure-mathematical approach to abstract categories as sets with morphisms (i.e., functions defined on the set's elements) and rules for composition of these morphisms. Categorical approach and algebras tend to assume that set, function and operation are primitive notions. These are our models of reality and that is the problem, because it is our mental image of the reality that is investigated by the pure mathematics, not the reality itself. Only groups and operations can actually be given in reality; everything else about them should be discovered. Often logical consequences of the pure-mathematical formalism diverge from the physical reality, as well as from the abstract reality that the PM itself creates by postulating its existence. The essence of my approach is a synthesis of concepts from consistent logical requirements and the contingencies posted on their actual existence. Postulated existence is arbitrary. It can result in entertaining nonexistent objects or denying existence to constructible ones. Only noncontradictory, constructible objects can actually exist. Existential postulates distort our perception of physical reality and bias our judgments.

PM defines also rings, ideals, modules and fields as generalized abstract algebraic structures [17]. The fields contain all four arithmetic operations, without the feared division by zero. The problem with this kind of abstract generalization is that they may create a nice illusion of completeness. The present paper was influenced by Elie Cartan's treatment of bilinear groups [18], but it goes much farther toward physics. For instead of defining group over a set, my question is: What should the set look like if it should permit quite unrestricted operations on all its elements? To answer it I will deploy the SM. PM is primarily devoted to properties of sets and views operations in terms of mappings, whereas SM explores abstract operational structures. In physical applications sets are just resources and mappings are regarded as tools. Operations are of primary interest to SM and their results to physics.

3. Need for an extra real zero, other than null

Being an additive group, the bigroup supports the self-evident relation:

$$0 + 0 = 0 \tag{0a}$$

which is correct for the natural zero (null). However, I just could not accept the almost 5000 years old prescription for multiplication of natural zeros:

$$0 \cdot 0 = 0 \tag{0b}$$

Henceforth the bold dot denotes multiplication of numbers. Comparing Eq. (0b) to the following well-known relation for the multiplicative unit:

$$1 \cdot 1 = 1 \tag{0c}$$

we see that Eq. (0b) is group-theoretically inadmissible, even though nobody else objected to it, as far as I know. It may be tolerated for natural numbers, but not for reals, unless it is understood as an approximation. For in Eq. (0b) natural zero acts with respect to itself just like the multiplicative unique unit in Eq. (0c). Eq. (0b) violates thus uniqueness of group operations, which must be unique for operational and structural reasons. It can still be used in practice as just an approximation, but not in no-nonsense algebraic theory.

The fact that PM tacitly disregarded this contradiction did not make it right. Uniqueness is absolutely necessary for operational consistency within each respective group and thus also within the bigroup. The prohibition of division by zero was a veiled consequence of the lack of uniqueness. The prohibition is effectively a postulative decree. Zero is an integer sometimes assumed as first natural number, even though it actually corresponds to an empty set or category 'null', or "nothing in there". Division requires real numbers to be fully implemented. Yet physics also needs pan-geometry [8] with multispatial hyperspace [7] whose mutually dual and twin member spaces should be convertible into each other. For this very reason, and also because unrestricted operations need closed system, the countable (integer) infinity ∞ would require its integer counterpart, which is the natural zero.

Whether one likes it or not, we need both zero and infinity to obtain an operationally closed number system. Because the natural zero belongs to integers and reals ($0 \in \Im \subset \Re$), the zero needs to be invertible under division too. Hence we must expand the reals by including the integer infinity $\infty \in \Upsilon$:

$$0 = 1/\infty \tag{0d}$$

which makes the incorrect Eq. (0b) only approximately true, since we get:

$$0 \cdot 0 = 1/\infty^2 \approx 0 \tag{0e}$$

where the result of multiplication of natural zeros does not really belong to integers, but to reals, of course. Elements of the set Υ can be called closing operands, if one has a problem with calling them numbers. The natural zero introduced as null also required a stretch of imagination to be called number.

Since there is nothing illogical in the above definition of bigroup and the nature evidently deploys bigroups anyway, then Eq. (0b) must be wrong. It was probably introduced to justify the prohibited division by zero, which was presumably made in order to just tacitly cover up our inability to define such an operation. PM quietly overlooked the inconsistency. The conflict must be resolved in order to operate on multispatial hyperspaces [7,8,10], which in turn is essential for comprehension of new physics [6,9]. There is more at theoretical stake than a squabble over multiplication of zeros, whose result is close to null anyway we count. Resolution of this conflict also calls for an extra zero in real (actually "irrational") numbers. The natural zero is an integer. We need also a real zero, because the natural numbers are of a different kind than irrational numbers, which can have somewhat indefinite value. We need also an indefinitely-valued, extra real zero that is different from null. This need is caused by the fact that the cardinal number $\aleph_0$ of the set of integers $\Im$ is smaller than the cardinal number $\aleph_1 = 2^{\aleph_0}$ of the set of the real numbers $\Re$. The set-theoretical issues shall be discussed elsewhere.

If consistent and operationally unrestricted number system with built-in division by zero must preserve symmetries inherent to all group operations, then it should contain an extra zero whose values with opposite signs should differ, just as for all the other (than natural zero) numbers, which come with their negative counterparts. This requirement is not really surprising to me, because it could post a natural foundation for the indeterminacy principle (IP) that emerged in quantum mechanics (QM). If the IP is valid as it surely is, then it should have already been implanted also in abstract mathematical infrastructure of physics. Utilizing synthetic approach to mathematics we should discover the infrastructure's unexpected abstract features, which we would probably never even dear to dream about in PM. Unlike mathematics, physics does not create facts, but uncovers or discovers them. The real zero supplies the IP with quite unambiguous mathematical underpinning, I think.

Although infinity is not a regular number, it plays the role of an inverse to zero. This very feature was symbolically expressed by rules of L'Hopital [19], which do apply to real numbers, and thus should be written as follows:

$$0_{\bullet} = 1/\infty_{\bullet} > 0 \iff 1/0_{\bullet} = \infty_{\bullet} \tag{1}$$

$$0_{\bullet} = 1/\infty_{\bullet} \iff 0_{\bullet} \cdot \infty_{\bullet} = 1 \tag{2a}$$

where the real infinity $\infty_{\bullet}$ and the real zero $0_{\bullet}$ are the limiting values. The real zero and real infinity are marked by bold dot following them. One can see that the real zero in Eq. (2) ceases to be just nothing and becomes "something". This real infinity is no longer the "everything" either. Real zero and real infinity provide thus operational closure to the multiplicative structure of the bigroup by extending the set $\Re$ of reals by two new operands so that $\infty_{\bullet}, 0_{\bullet} \in \Upsilon$. Eq. (2) also justifies the implied symbolic relation:

$$0 = 1/\infty \iff 0 \cdot \infty = 1 \tag{2b}$$

which could be surely called symbolic, because Eq. (2b) are not really constructible, due to the fact that the natural zero has null value and division operation cannot be fully implemented in integers. Although the rules of L'Hopital are also called symbolic, I agree that such a statement sounds like a lame excuse. I shall address it elsewhere. Eq. (2a) are constructible, however, because $0_{\bullet} > 0$ and also division can be fully implemented in reals. Nevertheless, one can see that Eqs. (2a) and (2b) are structurally similar.

Since anything divided by itself gives one, no matter what it is, we get:

$$0/0 = \infty/\infty = 0_{\bullet}/0_{\bullet} = \infty_{\bullet}/\infty_{\bullet} = x/x = 1 \tag{3}$$

for $x \in \{\Re \cup \Upsilon\}$. The real zero is just an infinitesimally small real (irrational) number. Both zeros and infinities are limiting values and therefore they can be understood in the same way as a limit is understood in calculus [20]. The natural zero was thought to be an integer: $0 \in \Im$. The set of integer numbers has been merged with rational and irrational numbers to form real numbers, but it is not really incorporated into the irrational numbers. Effectively thus, the real zero could be defined as the smallest physically meaningful positive irrational number at any given level of physics, but always greater than the natural zero, which is very complex and sophisticated theoretical concept. For length-based spaces (LBSs) the real zero could be defined in terms of Planck's length whereas the real infinity in terms of the corresponding to it Planck's scale energy. These values are not supposed to be absolutely fixed. As all irrational real numbers, they have essentially operational meaning.

When a real number is declared as an item to be stored in memory of a digital computer (as opposed to an analog one), no integers (except null-like zero) are explicitly represented among them. The number one emerges from rounding of the real number 0.999... or its particular binary representation on the given computer system. My point is that the natural zero, which has been imported from India some 5000 years ago, can serve as the unique null element of the additive group, whereas the real zero could be used merely as practically representable null operand in all multiplications that involve real numbers. One can assume that the real zero is used for calculations, while the natural zero is just its idealized, theoretical approximation. Their basic logical roles are distinct and their operational functions are quite different.

The ancient Indian abstract notion of natural zero is too great an idea to be wasted for ordinary calculations. It indicated absence of units in number systems, such as in 10 [21]. Zero was unique concept in operational sense too. In Sanskrit zero was called 'sunya' (empty) [22]. Babylonian zero sign did not yet signify the number zero or 'empty', but in Arabic, the imported from India zero (sifra) means 'empty' [23]. In ancient Far Eastern texts zero was not used. It was probably imported to Chinese mathematics from India in 7th century [24]. Although the natural zero has automated operations, the problem with incorporating it (and the number one) into group-theoretical structures had never been solved before. Most authors tacitly evaded those problems, whereas Weyl simply wrote: "Let us suppose that this has in some way been accomplished" [25], without addressing the whole issue at stake. Yet some hidden conceptual and structural conflicts with duality of abstract multispatial hyperspace, and group-theoretical approach to QM require that these issues be solved [7], even if nobody dears to complain. The present paper shows a possible approach and a quick solution to the whole problem. The solution needs not be sophisticated, but it must be operationally viable.

Zero is a powerful idea because it is infinity's twin [26]. Yet infinity as inverse to real zero is not something unattainable anymore. Let us examine few examples of other "sensitive" arithmetic operations that involve a real number $x > 0_{.}$ as well as the real zero and/or infinity as their other operand. I will use $x = 2$ or $x = 1/2$ in the examples of the new notions to be introduced:

$$\text{Superinfinite}: \quad x/0_{.} = x \cdot \infty_{.} = \infty_{.}^{\wedge} \text{ for } x > 1: \quad \text{example}: 2 \cdot \infty_{.} \tag{4a}$$

$$\text{Subinfinite}: \quad x/0_{.} = x \cdot \infty = \infty_{.}^{\sim} \text{ for } x < 1: \quad \text{example}: \infty_{.}/2 \tag{4b}$$

$$\text{Subinfinitesimal}: \quad x \cdot 0_{.} = x/\infty_{.} = \infty_{.\sim} \text{ for } x > 1: \quad \text{example}: 2/\infty_{.} \tag{4c}$$

$$\text{Superinfinitesimal}: \quad x \cdot 0_{.} = x/\infty_{.} = \infty_{.\wedge} \text{ for } x < 1: \quad \text{example}: 1/2 \cdot \infty_{.} \tag{4d}$$

The term $\infty_{.}^{\wedge}$ denotes a superinfinite number, a standian for a number from partition $(\infty_{.}, \infty_{.}^{2}]$. The term $\infty_{.}^{\sim}$ denotes a subinfinite number from partition $[1, \infty_{.})$. Their respective inverses are thus superinfinitesimal number from partition $(1/\infty_{.}, 1/\infty_{.}^{2})$, and subinfinitesimal number from partition $(1, 1/\infty_{.})$. We got actually two distinct types of infinities: rising (or pop-up) kind of infinity such as superinfinite and subinfinite, and shrinking (or drill-down) kind of infinity such as superinfinitesimal and subinfinitesimal. One could imagine that the subinfinite could be represented as bra-infinite $\langle \infty_{.}|$ while the superinfinite as ket-infinite $|\infty_{.}\rangle$, if one would not mind to "misuse" the splendid Dirac's symbols for such an ordinary purpose. In most practical applications, however, one may still assume that addition or multiplication with infinity as an operand also results in infinity, as set theory postulates. One could have noticed, however, that set theory overexposed some subtle nuances, while oversimplified others—mainly operational ones, which were of lesser interest to the PM. A very good point on that subject was made in [27]. The authors have explicitly noticed that the set of natural numbers (i.e. positive integers) is equipotent to a subset of its power set—with cardinality $\aleph_0$—but it is not equipotent to the power set itself, whose cardinal number is $\aleph_1 = 2^{\aleph_0}$, which is also the cardinal number of reals. This is the other than operational reason for my distinguishing of the real (irrational) infinity $\infty_{.}$ from the natural infinity ∞ of denumerable (i.e., infinitely countable) sets.

Hence infinity viewed as an operand is not the highest pseudo-number in an operational set, as our previous intuition might have deluded us. The infinity is an abstract turning point beyond which there may still sit some higher numbers, but we can only estimate them. Real zero is also such a turning point for the shrinking infinity that is called infinitesimal. Just as natural zero limits thus the measure of shrinking (though not any shrinking process as such), infinity too limits the measure of growing (though not any growth process as such). Thus both real zero and real infinity are infinities, or limits of definability. They did not really belong to the real numbers, even though the old natural zero was assumed as the initial natural number. Henceforth both zeros and infinities shall be treated as certain operational quasi-numbers, or as abstract operational switches, or just as turning points. We need them for quite unrestricted, abstract group-theoretical operability.

The subinfinite and subinfinitesimal numbers are essentially indefinite just as the real infinity is, from which they result, even though their expected value might lie well within our ability to estimate them, at least in principle. I shall discuss these issues in set-theoretical terms elsewhere. The speed of light in vacuum in relativity theories and Planck's constant in QM play the role of limiting physical constants in their respective theories. Although the real and the natural zero and both infinities are limiters, they do not really limit physically anything. They limit real numbers only operationally. The above relations (3) and (4) are intuitively quite clear. For one could sensibly divide zero by zero only once, just as every other number can be divided by itself, whereas anything other than zero could be divided by zero endlessly (infinitely long), for subtracting the natural zero (null) cannot decrease the original amount at all. There was nothing wrong with the natural infinity or natural zero, but the infamous by now prohibition of division by zero did more theoretical harm than good. The subtle natural zero had just too many conflicting theoretical and operational functions to perform all of them.

The real zero is uniquely equipped to work with fuzzy logic and also with many-valued logic in general. It was designed for multiplicative group. For addition the use of the real zero is not recommended, because one gets:

$$\text{for } x > 0: \quad x + 0_{.} = x + 1/\infty_{.} > x \tag{4e}$$

Addition works perfectly with the old, natural zero, which in its role as null was as if designed for the additive group. After all zero is the additive unit. Nonetheless, the real zero behaves much more like any other real number, because it has a definite though not sharply defined value. One can see that:

$$0_{.} - 0_{.} = 0, \quad 0_{.} + 0_{.} = 2/\infty_{.} > 0_{.} \tag{4f}$$

because the actual value assigned to the real zero should not be equal to null.

Now the mathematical infinity looks entirely different than its intuitive, philosophically spoiled twin. Multiplicative and additive group operations imply necessity for an extra real zero, which should possess certain nonnull value. Although the old natural zero is still the neutral unit of the additive group, we need also a real zero in order to construct the aforesaid bigroup. Do I have right to operate on infinity as if it were a number? Yes, I may try to operate on anything that fits the concept of group, as long as I can find a way to do it. No one should decide a priori what is not a number. Though our intuitions of numbers are not incorrect, this does not mean that what is called a numbers' set was quite completely defined by postulative decrees of pure mathematics. We should discover numbers' operational, structural and whatever else meaning they may possess, rather than just postulate it. Pure mathematics is not always true [7]—not even to itself, as I showed above.

Pure mathematics often cannot reconcile even its own concepts. This is because its creations were arbitrarily postulated. This is not just a problem of errors, which can and should be forgiven, but that of defiant disrespect for any fresh idea that is not found in encyclopedias or does not follow directly from Euclid & Company. As if Gödel did not prove that some truths cannot just be proved or disproved from whatever niveau has been attained thus far [28]. Therefore mathematical synthesis should enforce internal consistency of its objects by subjecting to experimental verification everything that we learned about them. Without such a synthetic scrutiny we are building yet another Almagest—a mathematical one this time. The PM is full of hidden logical inconsistencies [7,8], one of which endured for 5000 years, which is far longer than lifespan of most civilizations—superb achievement, indeed.

4. Another paradox buried deep in old mathematics

Evidently no conceptual problem is created by the once feared division by zero, except for the fact that it seems difficult to put the infinity on equal footing with other numbers. Eq. (0b) may have prompted the rejection of infinity as an operand of arithmetical operations, because it leads to very nasty logical paradox that was tacitly overlooked. Consider the relations:

$$0 \cdot 0 = 0 \Rightarrow 0 \cdot 1/\infty = 0 \Rightarrow 0 \cdot 1 = 0 \cdot \infty \Rightarrow 1(? = ?)\infty. \tag{5}$$

The relation (5), which can be derived from the infamous Eq. (0b), implies that the number one is the same as infinity. The nonsense identification of infinity with unity was interpreted as impossibility of operations on infinity, contrary to rules of L'Hopital, not that number one and infinity are identical. The old prescription for multiplication of zeros (0b) was the real cause of the paradox, which violates both mathematical and the common sense. The absurd was not induced by infinity, however, but by the fact that the natural zero was forced to play the double role of both the additive neutral unit and a virtual multiplicative unit at the same time. The two conflicting roles are irreconcilable, even though nobody ever challenged the abstract nonsense, as far as I can tell. But their hidden conflict did not really go away. Hence we must discard the postulated nonsense multiplication of zeros (0b) and to re-evaluate all the former conclusions that relied upon its presumed truth. The new prescription for multiplication of zeros (0e) avoids the paradox.

The once prohibited division by zero covered up the concealed paradox of relations (5). Hence we have no other rational choice, but must take Eq. (0e) instead of (0b), which undermines both the logical and the abstract algebraic infrastructure of former mathematics. Product of two real zeros, and consequently two infinities, should lead to a much deeper infinity than the regular infinity. Eq. (0b) is admissible only for the natural numbers, integers and as an approximation for the real numbers, but it is plain wrong in conceptual sense and invalid also in the group-theoretical sense. Eq. (0b) can only be used as an approximation for practical estimates.

Evidently Eq. (0b) is not a structural paradox of mathematics. When the postulate (0b) is dropped and infinity permitted as complementary, external inverse element to the real zero, then division by zero becomes admissible and no paradox emerges. In such a case the bigroup B would become quite unrestricted operationally. Whether one likes infinities or not, we have no other choice, but to admit that they complement the number systems just as the natural and real zero do. We do not have to understand infinity in order to conclude that it is group-theoretically necessary. To insist that the rule, which applies to all the other numbers, should be waived just for the natural zero would mean less than complete honesty [29] indeed.

5. Some operational roles for zeros and infinities

What kind of number is infinity anyway? Both zero and infinity enjoy very special operational status in nature. Theoretically the notion of natural infinity can be accommodated as a countable though uncounted yet number among the natural numbers [30]. But real numbers are not countable [31]. I shall deal with the infinities elsewhere. For operational purposes we do not need infinity to be computed, but as a turning point between dual LVSs [7]. Just as zero—when viewed as number rather than as a digit—is often used for sequencing and/or partitioning rather than calculations. Usually infinity is introduced axiomatically [32]. The paradox exposed by the relations (5) also demands that the issue of infinity must be effectively solved, rather than just covered up by the prohibition on division by zero. The idea of infinity is not just an obstacle, but also a blessing for operational aspects of abstract algebras. We could not fully operate without both infinities and both zeros.

Yet infinity can be thought of as just an abstract variable whose value can be made arbitrarily large, just as the old notion of an infinitesimally small increment has been replaced by arbitrarily small quasi-constant ε in modern calculus. Operationally speaking, the most important property of infinities is that they can serve as inverse elements to zeros, which makes quite unrestricted operations in real numbers' system feasible. The abstract algebraic field over reals can now serve as a model also for physical fields, which fact was tacitly assumed, but never actually investigated. It is clear that old preconceived philosophical connotations must not be attributed to advanced mathematical concepts, whose features must never be postulated, but preferably discovered and their predictions and consequences thereof should then be also experimentally verified, wherever it is possible.

Groups and other abstract algebraic structures were usually presented in an axiomatic form [11,17,33]. Based upon that, topological (or continuous) groups were introduced as abstract triples (G, m, t), where (G, m) is a group and (G, t) is a topological space: G is set, m is product (operation) and t is topology on G [34], with topology t on a set G defined as a collection of subsets of G such that sum and product of finite subsets (and null subset) belong to t [34, p. 87]. Topology also deployed groups for polygonal chains [35] or as chains of groups [36], but never explicitly, or perhaps never quite consciously, if you will, as simultaneous multigroups, even though it was an implicit assumption that multigroups would not raise any problems. In pure set-theoretical approach groups are treated as just sets and investigated only in terms of mappings [37]. Former mathematics was not really concerned with issues of operability, but merely with formal, abstract representations of algebraic structures. Therefore many inconsistencies still persist in it and are unchallenged; they were undetected for centuries or even millennia. The inconsistencies paralyzed mathematics [7] and often fooled physics [8,6,10].

According to a definition that dates back to Dedekind, rational number system is just an ordered quadruple $(R, \oplus, \otimes, \leqslant)$ such that it is an ordered field and it is isomorphic to its rational subfield. R is a set, $\oplus$ and $\otimes$ denote operations and $\leqslant$ is an ordering relation in R [38]. But irrational numbers were not defined as a class by their intrinsic properties, but just by negation (as those other than rational) [39]. However, unlike all natural and rational numbers, the irrational ones are dense [40] and this feature makes enormous difference. Properties of the set of reals were researched in a set-theoretical setting, but the numbers were not investigated thoroughly enough from an operational point of view. Yet properties of numbers and abstract algebraic structures are not confined to their respective theories.

In a subtle way they tell us something very important about the physical reality, even though PM used to disregard all that. We should reclaim mathematics from the art-like PM and pass those universal, abstract mathematical messages on to physics.

Galois found that no systematic procedure exists for finding the roots of higher than 4th degree equations [41]. This is not a message just for algebras or the group theory in particular. His was a universal message to whoever may listen that something really breaks down when we go up from abstract extension of three to four, just as when we jump from 3D to 4D structures in geometry. Yet the PM trampled over all this and even throws away its own theorems that support that suspicion [7]. Because irresponsible postulating of unwarranted properties is very simple and was never punished, the PM embraced it wholeheartedly. The supremacy of pure algebra over geometry started with Euler, who substituted trigonometric functions with series and thereby put algebra over geometry [42]. But the lack of synthetic approach that enforces built-in balance checks caused inconsistencies between algebra and number systems as well as geometry [7], not to mention physics [8,10]. The present note shows that we must correct deeply buried discrepancies, some of which bravely evaded scrutiny for 5000 years. The aforementioned requirement for unrestricted operations and invertible pairs of operands in particular, carry definite though rather abstract geometric meaning. But the question still remains open: could we really implement these requirements?

When approached from an abstract geometric point of view, the basic Eqs. (0d), (1), (2a) and (2b) depict mappings of a single point into an infinite multitude of other points and vice versa. Although such mappings may be difficult to imagine, they are conceivable in terms of Fourier analysis. I should recall that for all real numbers n and m such that $n \neq m$ we always obtain complex zero, and consequently thus also regular zero, from the following integral:

$$\int_0^T \exp i(n-m)\omega t \, dt = 0 + 0 \cdot i = 0 \tag{6}$$

whereas for $n = m$ the integral evaluates to T [43]. All these variables are the standard symbols used for evaluation of Fourier coefficients in analytic decomposition of complicated wave into multitude of simple sine/cosine waves: T is period (or rate of time), t is time-duration (or elapsed time), ω is an angular velocity, n and m are variable coefficients and i is the imaginary unit. This very sophisticated, unique mapping provides mechanism that can support an abstract inversion between zeros and infinities, and vice versa.

Also in projective geometry we got similar mapping, namely geometric projection of a 3D sphere onto an extended 2D plane that is tangent to the sphere. The plane's extension is a point at infinity [44,45]. The projective example could raise some questions about linear and planar infinities, which distinction has already been known to Jaina mathematics [46]—I shall deal with that elsewhere. Anyway, the mappings mean that the idea of having an extra real zero and an extra real infinity actually works both in theory and in practical applications. The mutually invertible functionality of zeros and infinities are thus feasible and can be implemented in geometry and in most analytic and physical applications. It can also support inversions between two distinct dual vector spaces. This aspect shall be discussed elsewhere.

6. Hyperdimensional geometry and Cantorian spacetime

I have indicated that physical reality resembles an abstract multispatial hyperspace composed of paired 3D simple geometric spaces, which can be identified with linear vector spaces (LVSs) for mathematical reasons [7] as well as for physical ones [10]. The proposal has distinct consequences that yield concrete predictions of entirely new facts as well as retrodictions of known facts [8]. These new predictions and retrodictions have already been confirmed directly in several formerly quite unexplained experiments and observations [6,9], and also indirectly hinted at in other valid experiments, which could not be evaluated by new mathematical formulas, because they require data that was not gathered back then when these experiments have been conducted [47–49]. Despite insufficient data, the indirect experiments revealed exactly the same trends that the new formulas predict [10,47]. Just as the regular equation of 2D plane in 3D space requires four independently varying components to be defined, the 3D spaces require fourth component to be linked together. As a matter of fact, I have already showed that the fourth component of the plane's equation spans two distinct 3D spaces [7]. This can be imagined as if one had removed fourth leg from all, except one four-legged chairs, and then chained the three-legged chairs so that fourth leg is always "supplied" by the next chair in the chain. This arrangement assures that every chair effectively rests on four legs, because each pair of the linear chain shares one leg. The two dual 3D spaces also appear as two 4D spacetimes. Yet the 4D spacetime is an abstract pseudo-spatial algebraic structure, not really simple geometric 4D space like the usual 3D space.

For many years I could not really understand the actual meaning of the curious result, which indicated that an abstract 4D spacetime must somehow be composed of two 3D dual LVSs and yet appear four-dimensional. Until I found the El Naschie's seminal paper [50], in which a way to fill spacetime with 4D spheres was proposed. Former physics used to accuse nature for concealing the badly needed extra dimensions, which modest unification of relativity with electromagnetism, and recently superstring theories required. Yet El Naschie suggested that perhaps fractal-like structure of an essentially infinite-dimensional Cantorian spacetime with complex time can provide a common geometric foundation for both quantum and classical physics [51]. Further inspiration was provided by a model where dimensions need not to be fixed, but may be dynamically determined [52], as well as by few other papers that shed light on the new interpretation of space and time in modern physics [53,54]. El Naschie showed that infinite-dimensional spacetime has physical relevance [55–69]—I shall discuss these issues elsewhere, after having expanded the ideas presented in this note in a set-theoretical setting.

El Naschie has envisioned the infinite-dimensional Cantorian spacetime with complex time as safe playground for physics, I think. His showing that abstract set-theoretical constructs make physical sense led me in pursuit of a new foundation for the abstract multispatial hyperspace that complies with experiments. The former formal approach to foundations of mathematics that was once celebrated as a way to prevent emergence of inconsistencies [70] failed to ensure paradox-free mathematical infrastructure. It is dead.

7. Summary and conclusions

Unrestricted operations on linear vector spaces require abstract algebraic bigroup that is both an additive and a multiplicative group at the same time with an extra real zero, which has certain not-null value, and with two quite distinct infinities: the regular natural infinity and a real infinity for the set of real numbers. The natural and real infinities are inverses of the natural and real zeros, respectively. The bigroup is an abstract operational structure for an infinite-dimensional, invertible Cantorian spacetime, which is preferred as the manifold for discrete multispatial hyperspace and for whole physics.

References

[1] Janiszewski Z. Sur le réalisme et l'idéalisme en mathématiques. In: Janiszewski Z, editor. Oeuvres choisies. Warszawa: PWN; 1962. p. 309–17.
[2] Klüver J. In: Operationalismus Kritik und Geschichte einer Philosophie der exakten Wissenschaften. Stuttgart: Problemata Frommann-Holzboog; 1971. p. 61.
[3] National Academy of Sciences: Physics in Perspective I. Washington, DC: NAS; 1972. p. 57.
[4] Rose HE, editor. Logic Colloquium'73. Amsterdam: North-Holland; 1975. p. 16.
[5] Bohr N. Can quantum-mechanical description of physical reality be considered complete. Phys Rev 1935;48:696–702.
[6] Czajko J. Intrinsic gravitational repulsion. Chaos, Solitons & Fractals 2004;20:683–700.
[7] Czajko J. Elie Cartan and Pan-Geometry of multispatial hyperspace. Chaos, Solitons & Fractals 2004;19:479–502.
[8] Czajko J. On conjugate complex time I: Complex time implies existence of tangential potential that can cause some equipotential effects of gravity. Chaos, Solitons & Fractals 2000;11:1983–92.
[9] Czajko J. On conjugate complex time II: Equipotential effect of gravity retrodicts differential and predicts apparent anomalous rotation of the sun. Chaos, Solitons & Fractals 2000;11:2001–16.
[10] Czajko J. On conjugate complex time III: Superstrings and complex Lorentz transformation. Chaos, Solitons & Fractals 2001;12:951–67.
[11] Kurosh AG. In: Lectures on general algebra. New York: Chelsea Publishing; 1965. p. 30ff.
[12] Bourbaki N. Eléments de mathématique: Groupes et algè bres de Lie. Paris: Hermann; 1960. p. 15.
[13] Banach S. Théorie des opérations linéaires, p. 13–219. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 42ff.
[14] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, p. 305–48. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 306.
[15] Banach S. Die Theorie der Operationen und ihre Bedeutung für die Analysis, p. 434–41. In: Banach S, editor. Oeuvres II. Warszawa: PWN; 1979. p. 436.
[16] Cartan E. Sur les groupes linéaires quaternioniensis, p. 251–63. In: Cartan E, editor. Oeuvres complètes. Paris: Gauthier-Villars; 1953.
[17] Waerden BL. van der, Algebra I. New York: Frederick Ungar Publishing; 1970. p. 32ff.
[18] Cartan E. Les groupes bilinéaires et les systèmes de nombres complexes, p. 7–10.5. In: Cartan E, editor. Oeuvres complètes, T1. Paris: Gauthier-Villars; 1953. p. 2.
[19] Stewart J. Calculus. Early Transcendentals. Pacific Grove: ITP; 1999. p. 305ff.
[20] Swokowski EW. Calculus. Boston: PWS Publishing; 1992. p. 40ff.
[21] Ifrah G. In: The universal history of numbers, vol. 245. New York: Wiley; 1998. p. 25.
[22] Menninger K. Number words and number symbols. A cultural history of numbers. Cambridge, MA: The MIT Press; 1969. p. 401.

[23] Ifrah G. In: From one to zero A universal history of numbers, vol 462. New York: Viking Penguin; 1985. p. 382.
[24] Mikami Y. The development of mathematics in China and Japan. New York: Chelsea Publishing; 1913. p. 83.
[25] Weyl H. Algebraic theory of numbers. Princeton, NJ: Princeton University Press; 1940. p. 33.
[26] Seife CZ. The biography of a dangerous idea. New York: Penguin; 2000. p. 2.
[27] Leung KT, Chen DLL. Elementary set theory. Hong Kong: Hong Kong University Press; 1992. p.125.
[28] Gödel K. On formally undecidable propositions of principia mathematica and related systems I. New York: Dover; 1992. p. 1, 40.
[29] Halmos PR. How to write mathematics, p. 157–86. In: Sarason DE, Gillman L, editors. Halmos–Selecta expository writing. New York: Springer-Verlag; 1983. p. 170ff.
[30] Singh J. Great ideas of modern mathematics: Their nature and use. New York: Dover Publications; 1959. p. 93.
[31] Lin S-YT, Lin Y-F. Set theory. An intuitive approach. Boston: Houghton-Mifflin; 1974. p. 85, 89.
[32] Suppes P. Axiomatic set theory. New York: Dover Publications; 1972. p. 138.
[33] Kurosh AG. The theory of groups II. New York: Chelsea Publishing; 1956. p. 233ff.
[34] McCarty G. Topology. An introduction. New York: McGraw-Hill; 1967. p. 111.
[35] Henle M. A combinatorial introduction to topology. New York: Dover Publications; 1994. p. 85ff.
[36] Hocking JG, Young GS. Topology. New York: Dover Publications; 1988. p. 243ff.
[37] Stoll RR. Set theory and logic. New York: Dover Publications; 1979. p. 346ff.
[38] Burrill CW. Foundations of real numbers. New York: McGraw-Hill; 1967. p. 118.
[39] James, James, editors. Mathematics dictionary. Princeton, NJ: D. Van Nostrand Co.; 1968. p. 203.
[40] Shapiro MS, editor. Mathematics encyclopedia. Garden City, NY: Doubleday & Co.; 1977. p. 105.
[41] Guillen M. Bridges to infinity. Los Angeles: Jeremy P. Tarcher, Inc.; 1983. p. 75.
[42] Grattan-Guiness I. The Norton history of mathematical sciences. New York: W.W. Norton & Co.; 1998. p. 300.
[43] Transnational College of LEX. Who is Fourier? Boston: Language Research Foundation; 2002. p. 367.
[44] Pedoe D. Geometry. A comprehensive course. New York: Dover Publications; 1998. p. 206.
[45] Brannan DA, Esplen MF, Gray JJ. Geometry. Cambridge: Cambridge University Press; 1998. p. 218.
[46] McLeish J. Number. New York: Fawcett Columbine; 1992. p. 118.
[47] Czajko J. Complex conjugate time in macro experiments. Chaos, Solitons & Fractals 2002;13:17–23.
[48] Czajko J. On the Hafele-Keating experiment. Ann Phys Leipzig 1990;47:517–8.
[49] Czajko J. Experiments with flying atomic clocks. Expt Tech Phys 1991;39:145–7.
[50] El Naschie MS. Hyperdimensional geometry and the nature of physical spacetime. Chaos, Solitons & Fractals 1999;10:155–8.
[51] El Naschie MS. Introduction to nonlinear dynamics, general relativity and the quantum. Chaos, Solitons & Fractals 1997;8:VII–X.
[52] Alvarez E, Cespedes J, Verdaguer E. On quantum dimension of space–time. Chaos, Solitons & Fractals 1994;4:411–4.
[53] El Naschie MS. On the irreducibility of spatial ambiguity in quantum physics. Chaos, Solitons & Fractals 1998;9:913–9.
[54] El Naschie MS. On conjugate complex time and information in relativistic quantum theory. Chaos, Solitons & Fractals 1995;5:1551–5.
[55] El Naschie MS. Radioactive decay and the structure of $E^{(00)}$ quantum spacetime. Chaos, Solitons & Fractals 1999;10:17–23.
[56] El Naschie MS. Renormalization semi-groups and the dimension of Cantorian spacetime. Chaos, Solitons & Fractals 1994;4:1141–5.
[57] El Naschie MS. A note on gravity and Cantorian spacetime. Chaos, Solitons & Fractals 1997;8:131–3.
[58] El Naschie MS. Dimensional symmetry breaking and gravity in Cantorian space. Chaos, Solitons & Fractals 1997;8:753–9.
[59] El Naschie MS. Fractal gravity and symmetry breaking in a hierarchical Cantorian space. Chaos, Solitons & Fractals 1997;8:1865–72.
[60] El Naschie MS. Penrose universe and Cantorian spacetime as a model for noncommutative quantum geometry. Chaos, Solitons & Fractals 1998;9:931–3.
[61] El Naschie MS. The fractal dimension of spacetime—remarks on theoretical derivation and experimental verification. Chaos, Solitons & Fractals 1998;9:1211–7.
[62] El Naschie MS. Knot complement with a three sphere volume as a model for $E^{(00)}$ spacetime. Chaos, Solitons & Fractals 1998;9:1787–8.
[63] El Naschie MS. Von Neumann geometry and $E^{(00)}$ quantum spacetime. Chaos, Solitons & Fractals 1998;9:2023–30.
[64] El Naschie MS. Superstrings, knots and noncommutative geometry in $E^{(00)}$ space. Int J Theor Phys 1998;37:2935–51.
[65] El Naschie MS. Jones' invariant, Cantorian geometry and quantum spacetime. Chaos, Solitons & Fractals 1999;10:1241–50.
[66] El Naschie MS. Quantum groups and hamiltonian sets on a nuclear spacetime Cantorian manifold. Chaos, Solitons & Fractals 1999;10:1251–6.
[67] El Naschie MS. On the unification of heterotic strings, M theory and $E^{(00)}$ theory. Chaos, Solitons & Fractals 2000;11:2397–408.
[68] El Naschie MS. Kleinian groups in $E^{(00)}$ and their connection to particle physics and cosmology. Chaos, Solitons & Fractals 2003;16:637–49.
[69] El Naschie MS. Modular groups in Cantorian $E^{(00)}$ high-energy physics. Chaos, Solitons & Fractals 2003;16:353–66.
[70] Gentzen G. Die gegenwärtige lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweis für die reine Zahlentheorie. Hildesheim: 1970. p. 13ff.