Spectral Graph Theory — Instructor: Padraic Bartlett

Homework 2

Week 3 — *Mathcamp 2011*

Attempt the problems that seem interesting! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard. Oh! Also, typos build character: if you find any (not that there ever could be such things in my problem sets,) correct them to the most reasonable thing you can think of and proceed from there!

1. In class, we found the eigenvectors/values of the directed cycle D_n. Use these to find the eigenvectors/values of the undirected cycle C_n.
2. Using our results on the cycle graph, find the spectrum of the path graph P_n.
3. Answer the question we asked yesterday on the HW/today in class: if G_1 and G_2 are a pair of graphs with the same spectrum, are G_1 and G_2 isomorphic?
4. Prove or disprove: There is no graph with eigenvalue -1/2.
5. $(-)$ What happens to a graph when you add additional vertices that aren't connected to anything?
6. $(-)$ In terms of the graphs G_1 and G_2, what's the spectrum of the graph given by the disjoint union of G_1 and G_2?
7. Find the spectrum of the Petersen graph. For extra style points, find it without ever actually looking at an adjacency matrix.

9. Prove the series of linear algebra propositions we stated in class, should you not believe them:
   (a) The area of a parallelogram spanned by the two vectors (a, b) and (c, d) is $|ad - bc|$; similarly, the area of a parallelepiped spanned by $(a, b, c), (d, e, f), (x, y, z)$ is $|aez - afy + bfx - bdz + cdy - cex|$.
   (b) If I_n is the $n \times n$ identity matrix, then $\det(I_n) = 1$.
   (c) Suppose that A is a $n \times n$ matrix. If A' is the matrix acquired by multiplying the k-th row of A by some constant λ, then $\det(A') = \lambda \det(A)$.
   (d) For any pair of $n \times n$ matrices A, B, $\det(AB) = \det(A) \cdot \det(B)$. In particular, this tells us that $\det(A^{-1}) = 1/\det(A)$, whenever A is an invertible matrix.